Swami Ramanand Teerth Marathwada University, Nanded Maharashtra, India.

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"Existence and Uniqueness of Solutions of Linear and Nonlinear Fractional Integro-Differential Equations"

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Introduction:

Fractional calculus is a field of mathematics that grows out of the traditional definitions of calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. The concept of fractional calculus (fractional derivatives and fractional integral) is not new. In 1695 L'Hospital asked the question as to the meaning of $\frac{d^n y}{dx^n}$ if $n = 1$; that is "what if n is fractional". Leibnitz replied that"$d^{\frac{1}{2}}x$ will be equal to $x\sqrt{dx} : x$."

The mathematical theory of fractional calculus was developed in the middle of the nineteenth century. It has been demonstrated that a lot has changed in the previous 100 years, particularly during the nineteenth century (Ross, 1977). Comparing fractional differential equations to classical differential equations, we show that they may be utilised to more accurately represent physical systems. Many studies have been done that have proven the benefits. In many dynamical systems, fractional calculus is utilised to model and control the behaviour. It is crucial to do research on approximation methods for fractional differential equations since exact solutions to fractional order differential equations are not always feasible.

Recently, fractional integrodifferential equations were investigated by many researchers in different problems, and a lot of papers were published in this matter (see, for example, [12, 14]). Furthermore, many boundary conditions were considered for the fractional-order integro-differential equations; some of these conditions are the classical, periodic, antiperiodic, nonlocal, multipoint, and the integral boundary conditions. Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint and nonlocal boundary value problems as special cases. Integral boundary value problems occur in the mathematical modelling of variety of physics processes and have recently received considerable attention.

Integrals and derivatives of functions are studied in fractional calculus. However, in this discipline of mathematics, we are interested in non-integer order integrals and derivatives rather than integer order integrals and derivatives. These are referred to as fractional integrals and derivatives, which can be of real or complex order, as a result, integer orders, are also included. Recently, the application of fractional calculus modelling is developed to several domains of research, including as rheology, biology, and geomorphology. Indeed, it is generally understood that the introduction of fractional derivatives provides the system with a critical memory mechanism. It is therefore important to understand the advantages of using fractional derivatives in classical equations. Currently, several different methods are used to solve fractional differential equations.

There are two basic definitions of fractional derivative[1, 2] which are

a) Riemann Liouville Fractional Derivative:

For $\alpha \in [n-1, n)$, the α-derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \frac{d^n}{dt^n} \int_a^{\alpha} \frac{f(x)}{(t-x)^{(\alpha-n+1)}} dx.$$

b) Caputo's Fractional Derivative:

For $\alpha \in [n-1, n)$, the α-derivativeof f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_a^{\alpha} \frac{f(x)}{(t-x)^{(\alpha-n+1)}} dx.$$

The Laplace transformation method can be used to solve Fractional Differential Equations[1, 3, 4]. The Laplace transform of Fractional derivative is given by,

a) Laplace Transform of Riemann Liouville Fractional Derivative

$$L\left\{{}_0^{\mathrm{a}}D_t^p f(t); s\right\} = s^p F(s) - \sum_{k=0}^{n-1} s^k \left[{}_0^{\mathrm{a}}D_t^{p-k-1} f(t)\right]_{t=0},$$

where $(n-1 < p \leq n)$

b) Laplace Transform of Caputo Fractional Derivative

$$L\left\{{}_0^{c}D_t^p f(t); s\right\} = s^p F(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0),$$

where $(n-1 < p \leq n)$.

Review of literature and recent developments in the subject area:

In 2009, Chang, Yong-Kui, V. Kavitha, and M. Mallika Arjunan[5], was proved some existence and uniqueness of mild solutions for a semi linear integro-differential equation of fractional order with nonlocal initial conditions in α-norm. Assume that the linear part generates a noncompact analytic semigroup. The results cover the cases that the nonlinearity F takes values in different spaces such as X, X_α and X_β, where $\alpha \leq \beta \in (0,1)$. Finally, some practical consequences are also obtained.

In 2009, Kosmatov, Nickolai[6], discussed the solvability of integral equations associated with initial value problems for a nonlinear differential equation of fractional order. The differential operator is the Caputo fractional derivative and the inhomogeneous term depends on the fractional derivative of lower orders. Obtain the existence of at least one solution for integral equations using the Leray–Schauder Nonlinear Alternative for several types of initial value problems. In addition, using the Banach contraction principle, they establish sufficient conditions for unique solutions. This approach in obtaining integral equations is the "reduction" of the fractional order of the integro-differential equations based on certain semigroup properties of the Caputo operator.

In 2010, Deng, Jiqin, and Lifeng Ma.[7], studied by using the fixed point theory, they discuss the existence and uniqueness of initial value problems for nonlinear fractional differential equations and obtain a new result.

In 2010, Bashir Ahmad, Ahmed Alsaedi[8], solved some initial value problem for a coupled Caputo type nonlinear fractional differential system of higher order. As a first problem, they exhibited that the non homogeneous terms in the coupled fractional differential system depend on the fractional derivatives of lower orders only. Their method of analysis is based on the reduction of the given system to an equivalent system of integral equations. Applying the nonlinear alternative of Leray-Schauder, they prove the existence of solutions of the fractional differential system. The uniqueness of solutions of the fractional differential system is established.

In 2013, Rahmat Darzi et al.[9], have investigate the existence and uniqueness of solution for fractional boundary value problem for nonlinear fractional differential equation with the integral boundary conditions where denotes Caputo

derivative of order by using the fixed point theory. They apply the contraction mapping principle and Krasnoselskii's fixed point theorem to obtain some new existence and uniqueness results.

In 2014, Yu, Tao, Ke Deng, and Maokang Luo.[10], the solvability of initial value problems for nonlinear Langevin equation involving two fractional orders are discussed in this paper. An existence result for the solution is obtained using the Leray–Schauder nonlinear alternative. In addition, sufficient conditions for unique solution are established under the Banach contraction principle. The existence results for the initial value problems of nonlinear classical Langevin equation follow as a special case of our results.

In 2014, Pholakrit Thiramaus, Sotiris K Ntouyas, Jessada Tariboon[11], have studied the existence and uniqueness of solutions for a fractional boundary value problem involving Hadamard-type fractional differential equations and non local fractional integral boundary conditions. The results are based on some classical fixed point theorems.

In 2016, Ahmad, Bashir, et al.[12], investigate a new kind of nonlocal multi-point boundary value problem of Caputo type sequential fractional integro-differential equations involving Riemann-Liouville integral boundary conditions. Several existence and uniqueness results are obtained via suitable fixed point theorems. Some illustrative examples are also presented.

In 2018, Hossein Fazli, Juan J Nieto, Fariba Bahrami[13], they proved the existence, uniqueness and smoothness of solutions for differential equations with Riemann-Liouville sequential fractional derivatives admitting only the existence of a lower solution. The appropriate fractional Sobolev spaces equipped with a suitable partial order are introduced, and applying some fixed point theorems the results are proved. The approximate solutions are presented to the corresponding initial value problems.

In 2019, Baleanu, D., et al.[14], One of the interesting fractional integro-differential equations is the three step crisis equation which has been reviewed recently. They investigate the existence of solutions for a three step crisis fractional integro-differential equation under some boundary conditions.

In 2020, Bragdi, Ahmed, Assia Frioui, and Assia Guezane Lakoud[15], have discussed of the Krasnoselskii fixed point theorem, the existence of solutions for a boundary value problem of nonlinear sequential fractional integro-differential equations are investigated. Two examples are given to illustrate our results.

In 2020, Devi, Amita, and Anoop Kumar[16], have investigate the existence and uniqueness(EU) of solutions for non-linear Langevin fractional differential equation (FDEs) in term of generalized Caputo fractional derivative(GCFD) of two distinct orders with periodic boundary conditions involving generalized fractional differential operator. The existence result is discussed by applied Krasnoselskii's fixed point theorem and the uniqueness of solution is derived by using Banach contraction mapping principle.

In 2021, Boulfoul, A., et al.[17], studied concerned with the existence and uniqueness of solutions for nonlinear fractional integro-differential equation on an unbounded domain in a weighted Banach space. New results are showed owing to the fixed point theorems of Banach and Krasnoselskii. Illustrate the results by examples to ending the theoretical study.

In 2022, Vidushi Gupta et al.[18], have studied the investigation of existence and uniqueness of impulsive dynamical fractional systems with quadratic perturbation of second type subject to non local boundary conditions is presented and proved. By employing the fractional theory, Banach contraction technique, and Krasnoselskii's fixed point theorem, they derived some sufficient conditions to ensure the existence of the system.

In 2023, El Mfadel, Ali, Said Melliani, and M'hamed Elomari[19], they have establish some new sufficient conditions for the existence of solutions to a class of boundary value problem for nonlinear fractional differential equations with Ψ-Caputo fractional derivatives of order $\beta \in (2, 3)$. The proofs are based on some fixed point theorems and some basic concepts of Ψ-fractional analysis. As application, several examples are given to illustrate the theoretical results.

On the other hand, the existence and uniqueness of solutions are among the most important qualitative properties of differential equations. The existence and

uniqueness of solutions of differential equations involving the fractional derivatives were tackled by many researchers (see [20–22] and the references therein).

In 2018, Y.Y. Gambo et al.[23], studied Cauchy problems for differential equations with the above derivative in the space of continuously differentiable functions. Nonlinear Volterra type integral equations of the second kind corresponding to the Cauchy problem are presented. Using Banach fixed point theorem, the existence and uniqueness of solution to the considered Cauchy problem is proved based on the results obtained

Objectives of proposed research:

1) To study basic concepts of calculus.
2) To study the fractional derivative and integrals.
3) To study the analytical solution of fractional differential equations.
4) To study some existence and uniqueness of initial value problems for nonlinear fractional differential equations
5) To derived some results on existence and uniqueness of solution for boundary value problems of fractional differential equations
6) To study the fractional integro-differential equations involving Riemann-Liouville and Caputo integral initial conditions.
7) To study the existence and uniqueness of solutions for nonlinear fractional integro-differential equations.

Research Methodology:

The following steps will be including in the methodology:

1) To refer the relevant reference & books on fractional differential equations.
2) To derived some new results existence and uniqueness for different fractional differential equations.
3) To check whether the solution of linear & nonlinear fractional integro-differential equations is exist or not and discuss the uniqueness of solution, if exist.
4) To study Mathematica, MATLAB, etc softwares to check the nature of solution graphically.

Importance of study:

In physics, engineering, biology, and economics, fractional calculus is becoming a valuable and important tool for analysing scientific problems. Fractional calculus, which involves the integration and differentiation of an arbitrary or fractional order, provides additional tools that extend the descriptive capacity of calculus beyond the classic integer-order concepts of rate of change and area under the curve.

International Status:

Many scientists and researchers many more contributed in the development of fractional differential equations and Fractional calculus like, Leibtniz, Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Grunwald, Riemann, Riesz, Miller, Ross, Igor Prodlubny, Hossein Fazli, Baleanu, D., Smith, Anderson, Roache, Lynn R. Willams (1979) Nguyen Huu Denn (1993).

National Status:

Indian's also contributed remarkable in the development of fractional calculus like R. K. Sexana, V.Srivastava, K. N. Rai, S. Das, K. Gupta, S. Saha Roy, V. B. L. Chawrasia, D. Kumar, Pradeep Kumar, Amita, and Anoop Kumar, Aalam, Suneel Kumar, Vidushi Gupta and Pant. S. P. Acharya worked on some results on fixed point in uniform space. S.L. Sing from Haridwar, R. P. Pant from Nainital, D.N Chaliszar from Gujrat, and many more from India had worked on fixed point theorems and its various applications. Regional Status: Many researchers from Maharashtra have been actively involved in the study of fractional calculus. Like M. Kolwankar, A. K. Pani, PSV Nataraj, S. Bhalekar, V. Daftardar- Gejji, J. Verma, D. B. Dhaigude, obtained common fixed point theorems in Dynamic programming.

Scope of research work:

1) Although a number of researchers have developed new operators involving several special functions, study of their properties, introduce alternate definitions

and generalizations of these operators but still there are several functions like I-function, Aleph function, Mittag-Leffler function etc.

2) The existence and uniqueness of solution for fractional boundary value problem for nonlinear fractional differential equation with the integral boundary conditions where denotes Caputo derivative of order by using the fixed point theory.

3) Investigate the existence and uniqueness (EU) of solutions for linear and non-linear different fractional differential equation (FDEs) in term of generalized Caputo fractional derivative (GCFD) of distinct orders with initial and boundary conditions involving generalized fractional differential operator.

Expected outcomes:

1. We will try to solve different FDE & find the solution of different fractional differential equations.
2. To study stability and convergence of the solutions graphically and analytically.
3. By analysing the solution of different FDE & we will try to derived statement of existence and uniqueness result of general FDE.
4. Repeated the procedure 1, 2 & 3 for different linear & non-linear fractional differential equations.
5. We will try to derived some results on existence and uniqueness for different fractional differential equations.
6. To find the approximate solutions of fractional differential equations by various Mathematical software.

Purpose of carrying out proposed research work:

The purpose of the research is to develop new iterative methods of fractional differential equations for obtaining solutions. The solution of fractional differential equations will be obtained by new methods using mathematical software. Also we will study existence and uniqueness results for different fractional differential equations and numerical methods to obtain the solutions of fractional partial differential equations. We will compare the solutions obtain by Analytical methods and Numerical methods. The results will be represented graphically.

Plan of work:

Sr. No.	Period	Plan of work	Targets to be Achieved
1	First Year	Study of background information relevant to the research issue of my research topic.	Excellent understanding of the subject and complete coursework
2	Second year	Collecting research articles related to the research topic.	Attempt to obtain new results based on existing work and submit some papers.
3	Third year	Publishing some papers in some reputed UGC care listed journals and writing of the thesis.	Published some more papers and thesis submission.

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